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Relativistic particles with rigidity generating non-standard examples of Willmore–Chen hypersurfaces

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Abstract

We study a natural extension to higher dimensions of the Nambu–Goto–Polyakov action. In particular, those dynamical objects evolving with $SO(3)$ symmetry in four dimensions. We show that this problem is strongly related to that of relativistic particles with rigidity of order three in a hyperbolic plane. The moduli space of solitonic solutions is completely determined in terms of the so-called rotation number. A quantization principle for closed solutions is also obtained and this gives a rational one-parameter family of Willmore–Chen hypersurfaces in the standard conformal structure of dimension four. Moreover, these are the first non-standard examples of this kind of hypersurfaces.

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1. Introduction

A particularly natural choice for the Lagrangian describing the dynamics in a bosonic string theory is the so-called Nambu–Goto action. It measures, up to a coupling constant, the area of the surfaces (worldsheets) in the ambient space. However, this theory presents serious difficulties, for example, it cannot be quantized. To overcome these troubles, one introduces the extrinsic curvature in the Lagrangian density. More precisely, one takes a QCD action in four dimensions which adds extrinsic curvature action to the usual Nambu–Goto area term. It has been set up by Polyakov [24] and independently by Kleinert [16]. In particular, the theory with extrinsic curvature action alone is very familiar to differential geometers. A variational problem associated with this action was formally introduced by Willmore in 1965 (see [28]). The so-called Willmore variational problem became popular for different reasons. First, the functional and so the associated theory are invariant under conformal changes in the background gravitational field. A second reason is the (still open) Willmore conjecture relative

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to surfaces of genus one [28]. The theory has been extended not only for hypersurfaces, but also for submanifolds in pseudo-Riemannian spaces (see, for example, [10–12, 21, 27, 29]). Therefore, the Willmore–Chen ($\mathcal{WC}$) submanifolds (in particular, the Willmore surfaces), that is the solutions to the field equations, are dynamical objects playing the role of branes in these theories [6, 7, 9]. There are many known examples of Willmore surfaces with constant mean curvature in spheres and also with non-constant mean curvature. The first known examples of $\mathcal{WC}$-submanifolds of dimension greater than 2 were obtained in [10]. There, the authors gave a one-parameter class of four-dimensional, $SO(4)$-invariant, $\mathcal{WC}$-submanifolds in the natural conformal structure on a round seven-sphere. More recently, the standard products of spheres which are $\mathcal{WC}$-hypersurfaces in $\mathbb{S}^{n+1}(\mathbf{1})$ have been determined [15]. They have constant mean curvature and are known as the *standard examples*. However, as far as we know, examples of $\mathcal{WC}$-hypersurfaces which do not fall within the conformal class of the standard examples are not known in the literature.

In this paper we exhibit the first non-standard examples of $\mathcal{WC}$-hypersurfaces. We consider the conformal $\mathcal{WC}$-action in the four-dimensional Euclidean space, acting on compact hypersurfaces. Then, we look for $SO(3)$-invariant compact solutions of the field equation. Next, we show that the reduced field equation coincides with that of relativistic spinning particles with a rigidity of order 3 on a hyperbolic plane. We are able to integrate, using the theory of elliptic functions, the field equation associated with this spinning particle and then to describe a moduli space of solutions. By exchanging the modulus defining this family of solutions, we introduce the so-called *rotation* in one period and show a quantization principle to describe the moduli subspace of closed solutions and then the rational one-parameter class of $SO(3)$-invariant $\mathcal{WC}$-hypersurfaces in this theory.

2. Symmetric Willmore–Chen hypersurfaces

In $\mathbb{R}^4$, we remove a certain straight line, say L. The remaining space $\mathbb{R}^4 - L$ can be identified, via an obvious diffeomorphism, with the product $\mathbb{H}^2 \times \mathbb{S}^2$, where $\mathbb{H}^2 = \{(u, v) \in \mathbb{R}^2/v > 0\}$ and $\mathbb{S}^2$ stands for a two-sphere. Let g_o be the Euclidean metric on the half-plane $\mathbb{H}^2$ and denote by $\mathrm{d}\sigma^2$ the radius one, round metric on $\mathbb{S}^2$. Then, the Euclidean metric, $\bar{g}_o$, on $\mathbb{R}^4 - L$ can be written as

$$\bar{g}_o = g_o + v^2\,\mathrm{d}\sigma^2.$$

In other words, the Euclidean space $\mathbb{R}^4 - L$ is nothing but the warped product $\mathbb{H}^2 \times_v \mathbb{S}^2$, where v is regarded as a positive smooth function, on the Euclidean half-plane, playing the role of warping function.

For any immersed curve $\gamma : [0, L] \to \mathbb{H}^2$, we have the hypersurface of $\mathbb{R}^4$, $\mathcal{T}_\gamma = \gamma \times_v \mathbb{S}^2$ and we will refer to $\mathcal{T}_\gamma$ as the tube around γ. Let $G = SO(3)$ be the group of isometries of $(\mathbb{S}^2, \mathrm{d}\sigma^2)$. Obviously, G acts transitively on $(\mathbb{S}^2, \mathrm{d}\sigma^2)$. We define an action of G on $\mathbb{R}^4 - L = \mathbb{H}^2 \times \mathbb{S}^2$ as follows:

$$A(\xi, p) = (\xi, A(p)) \qquad \text{for all} \quad (\xi, p) \in \mathbb{H}^2 \times \mathbb{S}^2 \quad \text{and} \quad A \in SO(3).$$

It is clear that this action is realized through isometries of $(\mathbb{R}^4 - L, \bar{g}_o)$. Moreover, the tubes around curves in $\mathbb{H}^2$ are G-invariant hypersurfaces. The following statement characterizes the tubes as the only hypersurfaces with $SO(3)$ gauge symmetry.

Proposition 2.1. *Let M be a G-invariant hypersurface of $(\mathbb{R}^4 - L, \bar{g}_o)$, then M is a tube around a certain curve in the half-plane $\mathbb{H}^2$.*

Proof. Since M is G-invariant, the orbit through every point, $(\xi, p) \in M$, is completely contained in M. On the other hand, G acts transitively on $\mathbb{S}^2$ and so the orbit through (ξ, p) is given by

$$[(\xi, p)] = \{(\xi, A(p))/A \in SO(3)\} = (\xi, \mathbb{S}^2).$$

This proves that M is foliated by two-spheres. The orthogonal distribution in M, being one dimensional, can be integrated to get a curve γ in $\mathbb{H}^2$ such that $\mathcal{T}_\gamma = \mathcal{M}$. □

Let $\mathcal{H}$ be the smooth manifold of compact hypersurfaces of $\mathbb{R}^4 - L$. It is clear that one has a natural action of G on $\mathcal{H}$ and the subset of symmetric points is $\mathcal{H}_G = \{\mathcal{T}_\gamma/\gamma$ is a curve immersed in $\mathbb{H}^2\}$. The Willmore–Chen functional [11], $\mathcal{WC} : \mathcal{H} \to \mathbb{R}$, is defined to be

$$\mathcal{WC}(M) = \int_M (\alpha^2 - \tau_e)^{\frac{3}{2}}\, \mathrm{d}v.$$

α and τ_e denote the mean curvature and the extrinsic scalar curvature functions of the hypersurface, respectively, and $\mathrm{d}v$ is the volume element associated with the induced metric on M. This functional is invariant under the above G-action. Moreover, it is known that the Lagrangian and the defined variational problem are invariant under conformal changes in the background metric [11]. Therefore, we can apply the principle of symmetric criticality here [20] to characterize those critical points that are G-invariant. These critical points are obtained as solutions of the so-called *reduced field equation*, that is the Euler–Lagrange equation of the restriction of the functional to $\mathcal{H}_G$ [2]. In other words, critical symmetric points are nothing but symmetric critical points.

To compute $\mathcal{WC}$ on $\mathcal{H}_G$, we take advantage of the above-mentioned conformal invariance. Then, we make the following conformal change in $(\mathbb{R}^4 - L, \bar{g}_o)$:

$$\bar{h}_0 = \frac{1}{v^2}\bar{g}_0 = \frac{1}{v^2}g_o + \mathrm{d}\sigma^2.$$

Now, we observe that $\left(\mathbb{H}^2, \frac{1}{v^2}g_o\right)$ is nothing but the hyperbolic plane with constant curvature -1. Therefore, we see that the new metric on $\mathbb{R}^4 - L$, which is conformal to the Euclidean one, is the Riemannian product of a hyperbolic plane with a round unit two-sphere. This fact can be used to prove that the extrinsic scalar curvature, τ_e, of any tube, vanishes identically. On the other hand, the mean curvature function, α, of a tube $\mathcal{T}_\gamma$, and the curvature function, κ, of the curve γ in the hyperbolic plane, $\left(\mathbb{H}^2, \frac{1}{v^2}g_o\right)$, are nicely related as follows (see [7] for general relationship):

$$\alpha^2 = \tfrac{1}{9}\kappa^2.$$

All this information can be joined to obtain the restriction of $\mathcal{WC}$ to the space of symmetric points

$$\mathcal{WC}(\mathcal{T}_\gamma) = \frac{4}{27}\pi \int_\gamma \kappa^3\, \mathrm{d}s.$$

As a consequence, we have the following result for the reduction of variables:

Theorem 2.2. *The tube $\mathcal{T}_\gamma$ is a Willmore–Chen hypersurface in $\mathbb{R}^4$ if and only if γ is a critical point of the following elastic energy action:*

$$\mathcal{L}_3(\gamma) = \int_\gamma \kappa^3\, \mathrm{d}s \tag{1}$$

which is assumed to act on closed immersed curves in $\mathbb{H}^2$ and where κ denotes the curvature function of γ in $\left(\mathbb{H}^2, \frac{1}{v^2}g_o\right)$.

Critical points of $\mathcal{L}_2(\gamma) = \int_\gamma \kappa^2\, ds$ are known as *elastic curves* or, simply, *elasticae*. The corresponding variational problem was introduced by Bernoulli and solved by Euler for curves in the Euclidean plane [26]. More recently, the study of *elasticae* in Riemannian manifolds has been a topic of intense study during the last years. To pick out just one example, one can consult the excellent paper of Langer and Singer [17], where they study the closed elastic curves in two-dimensional real space forms. A natural generalization of these curves are the *generalized elasticae or n-elasticae*. They are critical points of $\mathcal{L}_n(\gamma) = \int_\gamma \kappa^n\, ds$. Thus the above theorem can be rephrased as follows: *a tube $\mathcal{T}_\gamma$ is a Willmore–Chen hypersurface in $\mathbb{R}^4$ if and only if the base curve γ is a 3-elastic curve in the hyperbolic plane*, $\left(\mathbb{H}^2, \frac{1}{v^2} g_o\right)$.

Remark 2.3. Some remarks on the above result should be pointed out:

(i) The result gives an interesting characterization for $SO(3)$-invariant Willmore–Chen hypersurfaces in the four-dimensional Euclidean space. However, it does not prove the existence of such hypersurfaces. Existence will be shown later by exhibiting closed curves in the hyperbolic plane that solve the field equation associated with the Lagrangian $\mathcal{L}_3$.

(ii) Willmore–Chen tubes in $\mathbb{R}^4$ correspond to extended dynamical objects emerging when a round two-sphere propagates conformally (that is, moving without changing shape, only radius and position) in $\mathbb{R}^4$ along closed curves that are 3-elasticae in a hyperbolic plane. This hyperbolic plane describes the conformal factor.

(iii) $SO(3)$-invariant Willmore–Chen hypersurfaces can be interpreted as solitons of the conformal gravity on $\mathbb{R}^4$, whose energy travels as a localized packet. In this respect, these solitons have a particle-like behaviour. In fact, they are completely determined by partner, spinning massless relativistic, particles that evolve in a hyperbolic plane along 3-elastic trajectories. This constitutes a kind of holographic principle [18, 19, 22, 23].

3. 3-elasticae in the hyperbolic plane

In this section we study 3-elastic curves in $\left(\mathbb{H}^2, \frac{1}{v^2} g_o\right)$. Since we are interested in closed $\mathcal{WC}$-hypersurfaces, we will restrict ourselves to the space of closed curves. Otherwise, one might consider critical points that satisfy the given first-order boundary data, suitable to drop out the boundary terms which appear when computing the first-order variation of the action. To be precise, we consider the action that is defined by the Lagrangian $\mathcal{L}_3 : \Omega \to \mathbb{R}$, where Ω denotes the space of closed curves immersed in $\left(\mathbb{H}^2, \frac{1}{v^2} g_o\right)$. To compute the first-order variation of this action, we use a standard argument involving some integrations by parts (see, for example, [3, 19]), then we have

$$(\kappa^2)_{ss} + \tfrac{2}{3}\kappa^4 - \kappa^2 = 0 \tag{2}$$

where s and κ denote the arclength parameter and curvature of a curve, respectively. Therefore, if κ is constant then either $\kappa = 0$ or $\kappa = \sqrt{\frac{3}{2}}$. Having in mind the picture of curves with constant curvature in $\left(\mathbb{H}^2, \frac{1}{v^2} g_o\right)$, the former case corresponds with the geodesics which obviously are not closed while the latter one gives geodesic circles. These circles give rise to closed $\mathcal{WC}$-hypersurfaces in the four-dimensional Euclidean space with conformal constant mean curvature. The reader should note that after a well-known Aleksandrov theorem [1], the only embedded compact hypersurface with constant mean curvature in the four-dimensional Euclidean space are the round three-spheres. So, one could not expect solutions in $\mathbb{R}^4$, with constant mean curvature and non-trivial topology.

It will be useful to take $u(s) = \kappa^2(s)$ so that the Euler–Lagrange equation turns out to be

$$u_{ss} + \tfrac{2}{3}u^2 - u = 0$$

and so we multiply by u_s to obtain the following first integral:

$$u_s^2 = \tfrac{1}{9}(d - 4u^3 + 9u^2) = \tfrac{1}{9}Q(u) \tag{3}$$

where $d \in \mathbb{R}$ denotes a constant of integration. To integrate this equation, we note that Q is a third degree polynomial and so standard techniques in terms of elliptic functions can be used (see [13, 14] as general references). However, for this to make sense, we need $Q(u) \geqslant 0$ and $u = \kappa^2 \geqslant 0$. Since the polynomial function Q has a minimum in $u = 0$ with value $Q(0) = d$ and a maximum in $u = \frac{3}{2}$ with value $Q\left(\frac{3}{2}\right) = d + \frac{27}{4}$, then the above conditions hold if the minimum value is negative and the maximum one is positive. In other words when $d \in \left(-\frac{27}{4}, 0\right)$. In this case, for any value of d in this interval the polynomial function has three real roots which satisfy $\alpha_1^d < 0 < \alpha_2^d < \alpha_3^d < \frac{9}{4}$. Therefore, if we look for the solution, $u_d(s)$, with initial condition $u_d(0) = \alpha_2^d$, then using formulae 3.131 of [14], we see that it is a periodic function which is given by

$$u_d(s) = \kappa_d^2(s) = \frac{\alpha_1^d\left(\alpha_3^d - \alpha_2^d\right)\mathrm{sn}^2(p_d \cdot s, M_d) - \alpha_2^d\left(\alpha_3^d - \alpha_1^d\right)}{\left(\alpha_3^d - \alpha_2^d\right)\mathrm{sn}^2(p_d \cdot s, M_d) - \left(\alpha_3^d - \alpha_1^d\right)} \tag{4}$$

where $\mathrm{sn}(p_d \cdot s, M_d)$ denotes the *Jacobi elliptic sine* of modulus M_d, and where p_d, M_d are given by

$$M_d = \sqrt{\frac{\alpha_3^d - \alpha_2^d}{\alpha_3^d - \alpha_1^d}} \qquad \text{and} \qquad p_d = \frac{1}{3}\sqrt{\alpha_3^d - \alpha_1^d}. \tag{5}$$

The minima and the maxima of the above solutions are reached at

$$u_d(0) = \alpha_2^d \qquad \text{and} \qquad u_d\left(\frac{K(M_d)}{p_d}\right) = \alpha_3^d \tag{6}$$

where $K(M_d)$ is the complete elliptic integral of the first kind and modulus M_d.

All this information can be summarized in the following statement:

Proposition 3.1. *There exists a one-parameter family* $\left\{\gamma_d / d \in \left(-\frac{27}{4}, 0\right)\right\}$ *of 3-elastic curves with periodic curvature in the hyperbolic plane.*

Proof. Just define the curve γ_d in $\left(\mathbb{H}^2, \frac{1}{v^2}g_o\right)$ as that $\left(\text{up to isometries of } \left(\mathbb{H}^2, \frac{1}{v^2}g_o\right)\right)$ whose curvature function is given by κ_d, (4). □

It should be noted that the periodicity of the curvature function, κ_d, of a $\mathcal{L}_3$-critical curve, γ_d, is not enough to assure that γ_d is a closed curve.

4. Closed solutions and $\mathcal{WC}$-hypersurfaces

Our next goal is to determine the closed 3-elastic curves among those obtained before. We shall see that the parameter d can be exchanged for a new one with a deeper geometric meaning.

Let Ω be the smooth space of regular curves in $\mathbb{H}^2$. For $\gamma \in \Omega$ and $W \in T_\gamma\Omega$ (it can be viewed in $\mathbb{H}^2$ as a vector field along the curve), we define a curve $\alpha : (-\varepsilon, \varepsilon) \to \Omega, t \mapsto \alpha_t$, such that $\alpha_0 = \gamma$ and $\frac{\mathrm{d}\alpha_t}{\mathrm{d}t}\big|_{t=0} = W$. The picture in $\mathbb{H}^2$ is a variation of the initial curve along the variational field W. Denote by $w = |\alpha_t'|$ and κ the speed and the curvature functions, respectively, of the curves in the variation. A vector field W along a curve γ is called a *Killing field along* γ if $W(w) = W(\kappa) = 0$ [17]. If ∇ denotes the Levi–Civita connection on

$\left(\mathbb{H}^2, \frac{1}{v^2} g_o\right)$ and $\{T, N\}$ the Frenet frame along γ, then the Killing vector fields along γ are the solutions, X, of the following linear system (lemma 1.1 of [17]):

$$\begin{aligned} X(w) &= \langle \nabla_T X, T \rangle = 0 \\ X(\kappa) &= \left\langle \nabla_T^2 X - X, N \right\rangle = 0 \end{aligned} \tag{7}$$

where $\langle , \rangle = \frac{1}{v^2} g_o$ denotes the inner product in $\mathbb{H}^2$.

Obviously, the restriction of any Killing field of $\mathbb{H}^2$ to γ gives a Killing vector field along γ. The converse of this also holds [17]. Consequently, we have: $W \in T_\gamma \Omega$ is a Killing vector field along γ if and only if it extends to a Killing field (that will also be denoted by W) on $\left(\mathbb{H}^2, \frac{1}{v^2} g_o\right)$. Recall that a Killing field on the hyperbolic plane is called *translational* if it has an integral geodesic. If it has a unique zero, then it is called *rotational* while *horocyclical* means that it admits an integral curve being a horocycle. Then, we have

Proposition 4.1. *Let γ be a critical curve of the $\mathcal{L}_3$ Lagrangian on $\mathbb{H}^2$, included in the family described in the previous proposition, and let κ denote its curvature. Then the vector field defined by $J = 2\kappa^3 T + 6\kappa\kappa_s N$ is a rotational Killing field along γ.*

Proof. Since γ is a solution of the Euler–Lagrange equation for $\mathcal{L}_3$, its curvature function satisfies (2) and so $\nabla_T J = 3\kappa^2 N$ and $\nabla_T^2 J = -3\kappa^3 T + 6\kappa\kappa_s N$. Consequently, by using (7) we see that J is a Killing vector field along γ.

Now, take a vertex of γ, p_o, that is a critical point of the curvature function, κ. Denote by δ the integral curve of J through p_o. It is clear that δ has constant curvature, say $\widehat{\kappa_o}$. To compute $\widehat{\kappa_o}$, we first observe that δ is tangent to γ in the vertex p_o. Let ξ the unit tangent vector field to δ, then

$$\nabla_\xi \xi(p_o) = \nabla_{T(p_o)} \left(\frac{J}{\|J\|} \right).$$

Since p_o is a vertex and using (3) one can see that $\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\|J(s)\|}\right)$ vanishes at p_o and so

$$\nabla_\xi \xi(p_o) = \left(\frac{1}{\|J\|} (\nabla_T J) \right)(p_o) = \left(\frac{3}{2\kappa} N \right)(p_o).$$

Consequently, we have

$$\widehat{\kappa_o} = \frac{3}{2\kappa(p_o)}. \tag{8}$$

On the other hand, since d moves in $\left(-\frac{27}{4}, 0\right)$ then the greatest positive root of the polynomial Q varies in $\left(\frac{3}{2}, \frac{9}{4}\right)$, hence κ^2 satisfies $\alpha_2^d < \kappa_d^2 < \alpha_3^d < \frac{9}{4}$. Thus, we see from (8) that $\widehat{\kappa_o} > 1$. All this information shows that δ is a geodesic circle because it has constant curvature greater than 1 and this is enough to assure that J is rotational. □

The above proposition is the main point of the following argument. Let γ_d, $d \in \left(-\frac{27}{4}, 0\right)$, be a solution of the Euler–Lagrange equation for $\mathcal{L}_3$, and J be as given in the above proposition. We denote also by J its extension to a Killing field in the hyperbolic plane. Now, we choose a new, but equivalent, picture of $\left(\mathbb{H}^2, \frac{1}{v^2} g_o\right)$ which is adapted to the couple (γ_d, J). First we view the hyperbolic plane as the Poincaré disk of radius 1 and centred at the only zero of J. Then, we take polar coordinates, $x(\rho, \theta)$, in $\left(\mathbb{H}^2, \frac{1}{v^2} g_o\right)$ so that the curves of constant ρ are the integral curves of J, that is $\partial_\theta = b \cdot J$, for some $b \in \mathbb{R}$. In this coordinate system, we write $\gamma_d(s) = x(\rho(s), \theta(s))$. Then, we use the Euler–Lagrange equation to get

$$\theta_s = \frac{\langle T, \partial_\theta \rangle}{|\partial_\theta|^2} = \frac{2\kappa_d^3}{b\left(d + 9\kappa_d^4\right)} \tag{9}$$

and one may check that $b^2 d = -1$.

Let $h_d = \frac{2K(M_d)}{p_d}$ be the period of κ_d, where $K(M_d)$ is the complete elliptic integral of the first kind and modulus M_d, and where M_d, p_d are given as in (5). Then the rotation in one period of γ_d is given by

$$\Lambda_d = \sqrt{-d} \int_0^{h_d} \frac{2\kappa_d^3}{\left(d + 9\kappa_d^4\right)} \, \mathrm{d}s. \tag{10}$$

By a long computation that we sketch in the appendix, we obtain

$$\Lambda_d = \frac{2}{3} \frac{\sqrt{-d}}{n_d} \left\{ \frac{\alpha_2^d}{\tilde{q}_d} \Pi\left(\frac{\pi}{2}, \frac{m_d}{\tilde{q}_d}, \tilde{M}_d\right) - \frac{\alpha_2^d}{q_d} \Pi\left(\frac{\pi}{2}, \frac{m_d}{q_d}, \tilde{M}_d\right) \right\} \tag{11}$$

with $\Pi\left(\frac{\pi}{2}, \nu, \tilde{M}_d\right)$ being the complete elliptic integral of third kind and modulus $\tilde{M}_d$, and where

$$\tilde{M}_d = \sqrt{\frac{\alpha_1^d\left(\alpha_2^d - \alpha_3^d\right)}{\alpha_3^d\left(\alpha_2^d - \alpha_1^d\right)}} \qquad m_d = \frac{\sqrt{-d}\left(\alpha_3^d - \alpha_2^d\right)}{3\alpha_3^d} \tag{12}$$

$$q_d = \frac{\sqrt{-d}}{3} - \alpha_2^d \qquad \tilde{q}_d = \frac{\sqrt{-d}}{3} + \alpha_2^d \qquad n_d = \sqrt{\alpha_3^d\left(\alpha_2^d - \alpha_1^d\right)}. \tag{13}$$

From this, we can prove (see the appendix)

Proposition 4.2. *The rotation angle Λ_d, decreases (monotonically) from $\sqrt{2}\pi$ to π, as d moves from $-\frac{27}{4}$ to* 0.

Now, we would like to determine the closed 3-*elastic curves* among the above γ_d. Since $d \in \left(-\frac{27}{4}, 0\right)$, the curvature of $\gamma_d(s)$ is a periodic function, $\kappa_d(s)$, of period $h_d = \frac{2K(M_d)}{p_d}$. Moreover, note that since $\sinh(\rho(s)) = |\partial_\theta| = b|J|$, it follows from the expression of J given in proposition 4.1 that $\rho(s)$ is a periodic function whose period is a divisor of h_d. Hence, it is clear that if $\gamma_d(s)$ is a closed 3-*elastica*, then it closes up in an integer multiple of h_d and therefore its rotation in one period, Λ_d, must be a rational multiple of 2π. Conversely, if Λ_d is a rational multiple of 2π, then $\gamma_d(s)$ is closed. Hence, we have the following quantization principle: the rotation in one period of any closed 3-*elastica* comes only in rational multiples of some basic quantity of *charge*. More precisely,

Theorem 4.3. *Let γ_d, with $d \in \left(-\frac{27}{4}, 0\right)$ a* 3*-elastica with periodic curvature. Then γ_d is a closed curve in the hyperbolic plane if and only if its rotation in one period, Λ_d, is a rational multiple of 2π.*

Thus, using proposition 4.2, we have

Corollary 4.4. *For any couple of integers m, n such that $\frac{1}{2} < \frac{m}{n} < \frac{\sqrt{2}}{2}$, there exists a closed* 3*-elastica γ_{mn} in $\left(\mathbb{H}^2, \frac{1}{v^2} g_o\right)$.*

Let $d \in \left(-\frac{27}{4}, 0\right)$ be a real number for which $\Lambda_d = \frac{2n}{m}\pi$ as shown in the previous propositions. Let $\kappa_d(s)$ be the corresponding curvature functions and $\alpha_2^d, \alpha_3^d > 0$, the minimum and maximum values of $\kappa_d(s)$. Let us denote by $\gamma_d(s)$ the curve associated with

$\kappa_d(s)$ and by ϖ_1, ϖ_2 and ϖ_3, the circles of curvatures $\sqrt{\frac{3}{2}}$, $\frac{3}{2\sqrt{\alpha_2^d}}$ and $\frac{3}{2\sqrt{\alpha_3^d}}$, respectively. Then $\gamma_d(s)$ is a convex curve which oscillates between ϖ_2 and ϖ_3 and which closes up after m periods of $\kappa_d(s)$ and n trips around ϖ_1.

Finally, we have from theorem 2.2 and corollary 4.4,

Corollary 4.5. *There exists a rational one-parameter family, $\mathcal{T}_{\gamma_{mn}}$, $\frac{1}{2} < \frac{m}{n} < \frac{\sqrt{2}}{2}$, of closed Willmore–Chen hypersurfaces in the Euclidean four-space.*

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Appendix

Let γ_d be a solution of the $\mathcal{L}_3$-field equations corresponding to a value of d in the interval, $d \in \left(-\frac{27}{4}, 0\right)$. As before, we denote by α_1^d, α_2^d and α_3^d the three real roots of the polynomial $Q(u)$ given in (3). They satisfy $\alpha_1^d < 0 < \alpha_2^d < \alpha_3^d < \frac{9}{4}$.

Now, since $\kappa_d(s)$, the curvature of γ_d, is given by (4–6), then it is a periodic function of period $h_d = \frac{2K(M_d)}{p_d}$, where $K(M_d)$ is the complete elliptic integral of the first kind and modulus M_d (5). The function $\kappa_d(s)$ increases monotonically between its minimum and maximum, which are reached in $\kappa_d(0) = \sqrt{\alpha_2^d}$ and $\kappa_d\left(\frac{K(M_d)}{p_d}\right) = \sqrt{\alpha_3^d}$, respectively, and it is symmetric with respect to the line $y = \frac{K(M_d)}{p_d}$. Hence, using (3), we have

$$\Lambda_d = \sqrt{-d}\int_0^{h_d} \frac{2\kappa_d^3}{\left(d+9\kappa_d^4\right)}\,\mathrm{d}s = 6\sqrt{-d}\int_{\alpha_2^d}^{\alpha_3^d} \frac{u^2\,\mathrm{d}u}{(d+9u^2)\sqrt{u\left(u-\alpha_1^d\right)\left(u-\alpha_2^d\right)\left(\alpha_3^d-u\right)}}$$

which can be written as $\Lambda_d = I_1 + I_2$, where

$$I_1 = \frac{2}{3}\sqrt{-d}\int_{\alpha_2^d}^{\alpha_3^d} \frac{\mathrm{d}u}{\sqrt{u\left(u-\alpha_1^d\right)\left(u-\alpha_2^d\right)\left(\alpha_3^d-u\right)}}$$

and

$$I_2 = -\frac{2}{3}d\sqrt{-d}\int_{\alpha_2^d}^{\alpha_3^d} \frac{\mathrm{d}u}{(d+9u^2)\sqrt{u\left(u-\alpha_1^d\right)\left(u-\alpha_2^d\right)\left(\alpha_3^d-u\right)}}$$

Now, using 3.147 of [14] one gets

$$I_1 = \frac{4}{3}\frac{\sqrt{-d}}{n_d}K(\tilde{M}_d) \tag{A.1}$$

where n_d, $\tilde{M}_d$ are given in (12) and (13). Analogously, using 3.151 of [14], we have

$$I_2 = \frac{-4}{3}\frac{\sqrt{-d}}{n_d}K(\tilde{M}_d) + \frac{2}{3}\frac{\sqrt{-d}}{n_d}\left\{\frac{\alpha_2^d}{\tilde{q}_d}\Pi\left(\frac{\pi}{2}, \frac{m_d}{\tilde{q}_d}, \tilde{M}_d\right) - \frac{\alpha_2^d}{q_d}\Pi\left(\frac{\pi}{2}, \frac{m_d}{q_d}, \tilde{M}_d\right)\right\}. \tag{A.2}$$

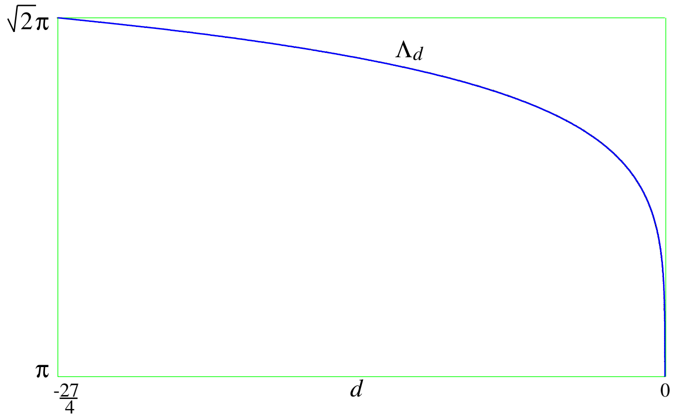

Figure A1. Variation of Λ_d as d moves in $(-\frac{27}{4}, 0)$.

Thus, from (A.1) and (A.2), we have

$$\Lambda_d = \frac{2}{3}\frac{\sqrt{-d}}{n_d}\left\{\frac{\alpha_2^d}{\tilde{q}_d}\Pi\left(\frac{\pi}{2}, \frac{m_d}{\tilde{q}_d}, \tilde{M}_d\right) - \frac{\alpha_2^d}{q_d}\Pi\left(\frac{\pi}{2}, \frac{m_d}{q_d}, \tilde{M}_d\right)\right\} \tag{A.3}$$

as we said.

Now, Λ_d can be seen as a real function which depends continuously on d. We want to compute the range of variation of Λ_d (see figure A1). By using the relations

$$\begin{aligned} d &= 4\left(\alpha_3^d\right)^3 - 9\left(\alpha_3^d\right)^2 \\ \alpha_1^d &= \frac{\left(9 - 4\alpha_3^d\right) - \sqrt{\left(9 - 4\alpha_3^d\right)\left(9 + 12\alpha_3^d\right)}}{8} \\ \alpha_2^d &= \frac{\left(9 - 4\alpha_3^d\right) + \sqrt{\left(9 - 4\alpha_3^d\right)\left(9 + 12\alpha_3^d\right)}}{8} \end{aligned} \tag{A.4}$$

and (12), (13), one can check that if $d \to -\frac{27}{4}$, then $\alpha_1^d \to \frac{-3}{4}$; $\alpha_2^d \to \frac{3}{2}$; $\alpha_3^d \to \frac{3}{2}$; $\tilde{M}_d \to 0$; $\frac{m_d}{q_d} \to 0$ and $\frac{m_d}{\tilde{q}_d} \to 0$. Since $\Pi\left(\frac{\pi}{2}, 0, 0\right) = \frac{\pi}{2}$, we have from (A.3) that $\lim_{d\to -\frac{27}{4}} \Lambda_d = \sqrt{2}\pi$.

Moreover, one can see that if $d \to 0$, then $\alpha_1^d \to 0$; $\alpha_2^d \to 0$; $\alpha_3^d \to \frac{9}{4}$; $\tilde{M}_d \to \frac{1}{\sqrt{2}}$; $\frac{m_d}{q_d} \to -\infty$ and $\frac{m_d}{\tilde{q}_d} \to \frac{1}{2}$. Hence the first term on the right-hand side of (A.3) goes to 0 as $d \to 0$. To compute the limit of the second term, we express it in terms of the Heuman's Lambda function $\mathbf{\Lambda}_o$. Denoting by $r_d = \frac{m_d}{\tilde{q}_d}$, we have

$$\Pi\left(\frac{\pi}{2}, r_d, \tilde{M}_d\right) = \frac{K(\tilde{M}_d)}{1 - r_d} + \frac{\pi}{2}\frac{r_d\left[\mathbf{\Lambda}_o(\beta, \tilde{M}_d) - 1\right]}{\sqrt{r_d(1 - r_d)\left(r_d - \tilde{M}_d^2\right)}} \tag{A.5}$$

where $K(\tilde{M}_d)$ is the complete elliptic integral of the first kind and modulus $\tilde{M}_d$ (12) and $\beta = \arcsin\frac{1}{\sqrt{1-r_d}}$. Now, using (12), (13), (A.4) and $\mathbf{\Lambda}_o(0, \tilde{M}_d) = 0$, one has $\lim_{d\to 0}\Lambda_d = \pi$. Therefore, Λ_d varies in $(\pi, \sqrt{2}\pi)$ as d moves in $\left(-\frac{27}{4}, 0\right)$.

In order to prove the monotonicity of Λ_d one might check the sign of its derivative with respect to d. This seems to be a big task in the light of the above computations. We have just checked it numerically. The computer generated the above graph of Λ_d in terms of d, when opted to process (10) and (4).

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